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Supersymmetric quantum mechanics and the index theorem

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We consider the classical mechanics of the spinning particle and investigate which *Abelian* interactions can be added without breaking supersymmetry. A quantum theory is presented. The well known index theorem for the Dirac operator is extended to take into account the effect of anti-symmetric Abelian tensor fields. Furthermore interactions with *non-Abelian* anti-symmetric tensor fields are investigated. It turns out in both cases that these fields do not give any non-trivial contributions to the index.

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1. Introduction

Supersymmetric quantum mechanics describes a Dirac particle that moves in d-dimensional space–time. This spinning particle model has been subject of much study in the context of the Atiyah–Singer index theorem. This theorem equates the index of an elliptic operator on a compact manifold to a topological invariant that characterizes the manifold. Alvarez-Gaumé showed [1] that a method to compute the index of an elliptic operator is obtained by finding a supersymmetric theory whose supercharge is exactly this operator. Then the index gives information about the spontaneous breaking of the supersymmetry of this theory. Alvarez-Gaumé and Witten described [2] the connection between the Atiyah–Singer index for a Dirac operator and the gravitational and gauge anomaly for fields with spin $\frac{1}{2}$ or spin $\frac{3}{2}$. This connection provides a method to calculate these anomalies in higher dimensions. For $d = 4k + 2$ they found another field that suffers from an anomaly. It is the anti-symmetric tensor field $A_{\mu_1 \ldots \mu_{2k}}$ whose field strength is self-dual. We also consider anti-symmetric tensor fields, but with an odd number of indices. They can be added to the spinning particle model without breaking supersymmetry. We calculate the index of the corresponding Dirac operator. Details of the calculations can be found in refs. [3,4].

2. The spinning particle

A relativistic point particle with spin $\frac{1}{2}$ in d dimensions can be described classically by an action

$$S = \int \mathrm{d}\tau \, [\tfrac{1}{2} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + \tfrac{1}{2} \mathrm{i} \psi^a \dot{\psi}^a - \tfrac{1}{2} \mathrm{i} \dot{x}^\mu \psi^a \psi^b \omega_\mu^{ab}(x)], \tag{1}$$

where $x^\mu(\tau)$ denotes the position of the particle as a function of its proper time τ. A dot denotes a derivative with respect to τ. The $\psi^a(\tau)$ are anti-commuting variables. They are the superpartners of $x^\mu(\tau)$ according to a global supersymmetry transformation generated by

$$Q = \psi^a e_a^\mu (p_\mu + \tfrac{1}{2} \mathrm{i} \omega_\mu^{bc} \psi^b \psi^c). \tag{2}$$

Indeed if the basic Dirac brackets are given by

$$\{x^\mu, p_\nu\} = \delta_\nu^\mu, \qquad \{\psi^a, \psi^b\} = -\mathrm{i}\eta^{ab}, \tag{3}$$

the supersymmetry transformations are

$$\begin{aligned} \delta_\epsilon x^\mu &= \mathrm{i}\{\epsilon Q, x^\mu\} = -\mathrm{i}\epsilon \, e_a^\mu \, \psi^a \equiv -\mathrm{i}\epsilon \, \psi^\mu, \\ \delta_\epsilon \psi^\mu &= \mathrm{i}\{\epsilon Q, \psi^\mu\} = \epsilon \dot{x}^\mu. \end{aligned} \tag{4}$$

The supercharge Q satisfies the Dirac bracket relation

$$\{Q, Q\} = -2\mathrm{i}H, \tag{5}$$

where H is the Hamiltonian, the generator of proper time translations. Another conserved charge is

$$\Gamma^* = -\frac{\mathrm{i}^{\lfloor 5d/2 \rfloor}}{d!} \epsilon_{a_1 \ldots a_d} \psi^{a_1} \ldots \psi^{a_d}, \tag{6}$$

which is the generator of duality transformations on ψ^a. More symmetries of the spinning particle can be found in ref. [5].

Quantization now consists of replacing phase space variables by Hermitian operators whose (anti-)commutators are i times the Dirac brackets (3). The operators $\widehat{x}^\mu$ and $\widehat{p}_\mu$ are represented in the standard way and the variables ψ^a are represented by γ-matrices

$$\widehat{\psi}^a = \frac{1}{\sqrt{2}} \gamma^a. \tag{7}$$

Hence after quantization the second of eq. (3) reduces to the Clifford algebra. Quantization of the supercharge gives the Dirac operator

$$\mathrm{i}\gamma \cdot D = \mathrm{i}\gamma^c e_c^\mu \left(\partial_\mu - \tfrac{1}{4} \omega_\mu^{ab} \gamma^a \gamma^b\right). \tag{8}$$

The quantum version of the dual charge is the generator of chiral transformations in d dimensions:

$$\Gamma^* = -\frac{\mathrm{i}^{\lfloor 5d/2 \rfloor}}{2^{d/2} d!} \epsilon_{a_1 \ldots a_d} \gamma^{a_1} \ldots \gamma^{a_d} = \frac{1}{2^{d/2}} \gamma_{d+1}, \tag{9}$$

with

$$\gamma_{d+1}^2 = 1. \tag{10}$$

Hence the eigenvalues of γ_{d+1} are $+1$ and -1, and the eigenstates are spinors with positive and negative chirality,

$$\gamma_{d+1}\Phi_\pm = \pm\Phi_\pm. \tag{11}$$

Since H and γ_{d+1} commute, they have a common set of eigenstates:

$$H\Phi_\pm = \epsilon_\pm\Phi_\pm. \tag{12}$$

The supercharge Q anti-commutes with γ_{d+1} but commutes with H, which means that it changes the chirality of a chiral spinor, but not its energy. The result is, that for a non-zero energy eigenvalue ϵ there is a pair of eigenstates with opposite chirality

$$Q\Phi_\pm = \sqrt{\epsilon}\Phi_\mp \Rightarrow H\Phi_\pm = Q^2\Phi_\pm = \epsilon\Phi_\pm. \tag{13}$$

In the case when $\epsilon = 0$ this does not hold. Since $H = Q^2$ we find

$$H\Phi = 0 \quad \Leftrightarrow \quad Q\Phi = 0. \tag{14}$$

Hence the Witten index, which is defined as the difference between the number n_+ of positive and n_- of negative chirality states:

$$\Delta = n_+ - n_- \tag{15}$$

may get contributions from the zero modes. If Φ_+ is a boson then Φ_- is a fermion, hence

$$\Delta = \mathrm{Tr}(-)^F \, \mathrm{e}^{-\beta H}. \tag{16}$$

The regulator $\mathrm{e}^{-\beta H}$ does not give any contribution for only states with $H = 0$ contribute. This implies that the expression is independent of β. Alvarez-Gaumé showed that the index can be written as a path-integral

$$\Delta = \int_{\text{P.B.C.}} \mathrm{d}x\,\mathrm{d}\psi\,\exp\left(-\int_0^\beta L(x,\psi)\right). \tag{17}$$

Calculation of this path-integral gives the well-known result

$$\Delta = \frac{1}{(2\pi)^{1/2}} \int_{\text{space}} \left[\det\left(\frac{\mathrm{i}R_{ab}/2}{\sinh(\mathrm{i}R_{ab}/2)}\right)\right]^{1/2}, \tag{18}$$

with

$$R_{ab} = R_{ab\mu\nu}(\omega_\kappa^{cd})\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \tag{19}$$

where $R_{ab\mu\nu}$ is the Riemann curvature tensor. In order to find an expression for the index in d dimensions we only pick out the term in the expansion of the integrand that contains a d-form. This expression is indeed β-independent.

3. Interactions

The most general supersymmetric action which is at most quadratic in time derivatives and which only contains bosonic background fields is given by

$$
\begin{aligned}
S = \int \mathrm{d}\tau \sum_{\alpha=0}^{\lfloor (d-1)/2 \rfloor} \Big(& \frac{(-\mathrm{i})^{\alpha}}{(2\alpha)!} \dot{x}^{\mu} \psi^{\mu_1} \cdots \psi^{\mu_{2\alpha}} A_{\mu\mu_1\cdots\mu_{2\alpha}}(x) \\
& + \frac{(-\mathrm{i})^{\alpha+1}}{(2\alpha+2)!} \psi^{\mu_1} \cdots \psi^{\mu_{2\alpha+2}} F_{\mu_1\cdots\mu_{2\alpha+2}}(x) \\
& + \frac{(-\mathrm{i})^{\alpha+1}}{2(2\alpha+1)!} \dot{\psi}^{\mu} \psi^{\mu_1} \cdots \psi^{\mu_{2\alpha+1}} g_{\mu\mu_1\cdots\mu_{2\alpha+1}}(x) \\
& + \frac{(-\mathrm{i})^{\alpha}}{2(2\alpha)!} \dot{x}^{\mu} \dot{x}^{\nu} \psi^{\mu_1} \cdots \psi^{\mu_{2\alpha}} g_{\mu\nu\mu_1\cdots\mu_{2\alpha}}(x) \\
& + \frac{(-\mathrm{i})^{\alpha+1}}{(2\alpha+2)!} \dot{x}^{\mu} \psi^{\mu_1} \cdots \psi^{\mu_{2\alpha+2}} \Gamma_{\mu\mu_1\cdots\mu_{2\alpha+2}}(x) \Big) ,
\end{aligned}
\tag{20}
$$

with, for $\alpha = 0, \cdots, \lfloor (d-1)/2 \rfloor$

$$
F_{\mu_1\cdots\mu_{2\alpha+2}} = (2\alpha+2)\partial_{[\mu_1} A_{\mu_2\cdots\mu_{2\alpha+2}]}, \tag{21}
$$

$$
\begin{aligned}
\Gamma_{\mu\mu_1\cdots\mu_{2\alpha+2}} = & -\tfrac{1}{2}\{ g_{\mu\mu_1\cdots\mu_{2\alpha+1},\mu_{2\alpha+2}} - g_{\mu_{2\alpha+2}\mu_1\cdots\mu_{2\alpha+1},\mu} \\
& - g_{\mu\mu_{2\alpha+2}\mu_2\cdots\mu_{2\alpha+1},\mu_1} - \cdots - g_{\mu\mu_1\cdots\mu_{2\alpha}\mu_{2\alpha+2},\mu_{2\alpha+1}} \}.
\end{aligned}
\tag{22}
$$

The field $A_{\mu_1\cdots\mu_{2\alpha+1}}$ is anti-symmetric in all its indices, while $g_{\mu\mu_1\cdots\mu_{2\alpha+1}}$ is anti-symmetric in all indices except the first one. Furthermore, without loss of generality, we can take

$$
g_{[\mu\mu_1\cdots\mu_{2\alpha+1}]} = 0. \tag{23}
$$

This justifies our interpretation of $g_{\mu\nu}$ as a metric. The case when $\alpha = 0$ describes a spinning particle in a gravitational and electromagnetic field [6]. Notice that S is invariant under a gauge transformation of the form

$$
A_\mu \to A_\mu + \partial_\mu \Lambda. \tag{24}
$$

The anti-symmetric tensor fields do not transform under any gauge transformation. We call the generalized connections torsionless because

$$
\Gamma_{[\mu\mu_1\cdots\mu_{2\alpha+1}]\mu_{2\alpha+2}} = 0, \tag{25}
$$

and we find that the fields $A_{\mu\mu_1\cdots\mu_{2\alpha+2}}$ can be interpreted as the torsions added to the connections (24).

The new interaction terms do not spoil the dual symmetry that was mentioned before. The canonical quantization procedure can be done in a way analogous to the case without the extra interaction terms. Details of this calculation can be found in ref. [3]. From now on we take the generalizations of the metric $g_{\mu\mu_1\cdots\mu_{2\alpha+1}}$, $(\alpha > 0)$ to vanish.

The Dirac operator corresponding to (20) contains contributions from the A-fields:

$$\mathrm{i}\gamma \cdot D = \mathrm{i}\gamma^c e_c^\mu \left(\partial_\mu - \tfrac{1}{4}\omega_\mu^{ab}\gamma^a\gamma^b + \sum_{\alpha=0}^{\lfloor (d-1)/2 \rfloor} \frac{(-\mathrm{i})^{\alpha+1}}{2^\alpha (2\alpha+1)!} \gamma^{a_1} \cdots \gamma^{a_{2\alpha}} A_\mu^{a_1 \cdots a_{2\alpha}} \right), \tag{26}$$

where the curved indices have been transformed into flat indices by contraction with (inverse) vielbeins. We have obtained the index of this Dirac operator in the special case that the A-fields are exact:

$$A_{\mu_1 \cdots \mu_{2\alpha+1}} = (2\alpha + 1)\partial_{[\mu_1} A_{\mu_2 \cdots \mu_{2\alpha+1}]}, \tag{27}$$

by calculating the path-integral (17) for the Lagrangian corresponding to the action (20). The expression that we find is of the form (18) with R_{ab} replaced by a torsion dependent tensor R^{T}_{ab}:

$$R^{\mathrm{T}}_{ab}(\omega_\kappa^{cd}) = R_{ab}\left(\omega_\kappa^{cd} - \sum_{\alpha=0}^{\lfloor (d-3)/2 \rfloor} \frac{(-\mathrm{i})^\alpha}{(2\alpha+1)!} \beta^{-\alpha} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_{2\alpha}} A^{cd}_{\kappa\mu_1 \cdots \mu_{2\alpha}} \right). \tag{28}$$

It seems that the expression for the index is β-dependent because of the β-dependence of R^{T}_{ab}. However, we can show that expanding (18) with R^{T}_{ab} of the form (28) the contribution from the torsion fields only consist of globally defined exact forms [3]. Thus if one considers field configurations which vanish at the boundary of the manifold, the A-fields do not contribute to the index, which is again given by (18).

4. Non-Abelian tensor fields

Except for the one index tensor field A_μ, the anti-symmetric $(2\alpha + 1)$-tensors that we discussed in the previous section did not have any gauge invariance. We now introduce anti-symmetric tensor fields in a gauge invariant way, following ref. [7]. We add two canonically conjugate sets of fermion fields $\eta^m(\tau)$ and $\bar{\eta}^m(\tau)$, $(m = 1, \ldots, D)$ by adding to the action (1) a term

$$\Delta S = \int \mathrm{d}\tau \, (-\mathrm{i}\bar{\eta}^m \dot{\eta}^m). \tag{29}$$

This term is invariant under a global transformation

$$\bar{\eta}^m \to (\bar{\eta}\, \mathrm{e}^{\Lambda})^m, \qquad \eta^m \to (\mathrm{e}^{-\Lambda}\eta)^m, \tag{30}$$

with Λ^{mn} some constant matrix. If we make this symmetry local by taking Λ^{mn} x-dependent, we should introduce a covariant derivative containing a gauge field that we may take ψ-dependent. Instead of (29) we now take

$$\begin{aligned}\Delta S &= \int \mathrm{d}\tau\,[-\mathrm{i}\bar{\eta}^m(\mathrm{D}_\tau\eta^m)] \\ &= \int \mathrm{d}\tau\left[-\mathrm{i}\bar{\eta}^m\dot{\eta}^m + \mathrm{i}\bar{\eta}^m\sum_{\alpha=0}^{\lfloor(d-1)/2\rfloor}\left(\frac{(-\mathrm{i})^\alpha}{(2\alpha)!}\dot{x}^\mu\psi^{\mu_1}\cdots\psi^{\mu_{2\alpha}}A_{\mu\mu_1\cdots\mu_{2\alpha}}^{\ mn}(x)\right.\right. \\ &\quad \left.\left.+\frac{(-\mathrm{i})^{\alpha+1}}{(2\alpha+2)!}\psi^{\mu_1}\cdots\psi^{\mu_{2\alpha+2}}F^{mn}_{\ \mu_1\cdots\mu_{2\alpha+2}}(x)\right)\eta^n\right]. \end{aligned} \tag{31}$$

Notice that the indices m and n are group indices, and not flat indices. If we define $(2\alpha+1)$-forms and $(2\alpha+2)$-forms

$$\begin{aligned} A^{(2\alpha+1)} &\equiv \frac{(-\mathrm{i})^\alpha}{(2\alpha+1)!}\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_{2\alpha+1}}A_{\mu_1\cdots\mu_{2\alpha+1}}, \\ F^{(2\alpha+2)} &\equiv \frac{(-\mathrm{i})^\alpha}{(2\alpha+2)!}\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_{2\alpha+2}}F_{\mu_1\cdots\mu_{2\alpha+2}}. \end{aligned} \tag{32}$$

Then the field strength in (31) is defined by

$$F^{(2\alpha+2)} = \mathrm{d}A^{(2\alpha+1)} + \sum_{\beta=0}^{\alpha} A^{(2\beta+1)}\wedge A^{(2\alpha-2\beta+1)}. \tag{33}$$

The action $S+\Delta S$ as defined by (1) and (31) is symmetric under a supersymmetry that now also involves η^m and $\bar{\eta}^m$. It is also invariant under the gauge transformations

$$\begin{aligned} A^{(2\alpha+1)} &\to \Phi A^{(2\alpha+1)}\Phi^{-1} + \delta_{\alpha,0}\Phi\,\mathrm{d}\Phi^{-1}, \\ F^{(2\alpha+2)} &\to \Phi F^{(2\alpha+2)}\Phi^{-1}. \end{aligned} \tag{34}$$

Notice that only the transformation rule for $A^{(1)}$ contains an inhomogeneous term.

The Dirac operator that corresponds to this theory is defined by

$$\begin{aligned}\mathrm{i}\gamma\cdot D &= \mathrm{i}\gamma^c e_c^\mu\Big(\partial_\mu - \tfrac{1}{4}\omega_\mu^{ab}\gamma^a\gamma^b \\ &\quad - \mathrm{i}\hat{\bar{\eta}}^m\sum_{\alpha=0}^{\lfloor(d-1)/2\rfloor}\frac{\overline{(-\mathrm{i})^{\alpha+1}}}{2^\alpha(2\alpha+1)!}\gamma^{a_1}\cdots\gamma^{a_{2\alpha}}A_{\mu a_1\cdots a_{2\alpha}}^{mn}\hat{\eta}^n\Big). \end{aligned} \tag{35}$$

with $\hat{\eta}^m$ and $\hat{\bar{\eta}}^n$ represented in terms of Hermitian $2^D\times 2^D$ Dirac matrices. The Witten index of this Dirac operator is found to be of the form

$$\Delta = \frac{1}{(2\pi)^{d/2}}\int_{\text{space}} \mathrm{Tr}\exp\mathrm{i}\hat{F}\left[\det\left(\frac{iR_{ab}/2}{\sinh(iR_{ab}/2)}\right)\right]^{\frac{1}{2}}. \tag{36}$$

with

$$\hat{A} = \sum_{\alpha=0}^{\lfloor(d-1)/2\rfloor}\beta^{-\alpha}A^{(2\alpha+1)};$$

$$\widehat{F} = \mathrm{d}\widehat{A} + \widehat{A} \wedge \widehat{A} = \sum_{\alpha=0}^{\lfloor (d-1)/2 \rfloor} \beta^{-\alpha} F^{(2\alpha+2)}. \tag{37}$$

Again we find a β-dependent expression for the index. However, we will show that the terms that depend on β only give a trivial contribution, as in the case of the generalized torsion tensors described in the previous section. Writing out (36), it is found that the index is given by an integral over terms of the form

$$\mathrm{Tr}(\mathrm{i}\widehat{F})^p \mathcal{F}(R), \tag{38}$$

with $\mathcal{F}(R)$ a closed function of R: $\mathrm{d}\mathcal{F}(R) = 0$. It is well known that $\mathrm{Tr}\, F^p$ can locally be written as the exterior derivative of a Chern–Simons form. This cannot be done globally because on the overlapping region of two coordinate patches one may get contributions from the inhomogeneous term in (34) (winding numbers). Analogously $Tr\widehat{F}^p$ can be locally written as the exterior derivative of some generalized Chern–Simons form. However, only for $\alpha = 0$ there is an inhomogeneous term in the gauge transformation rule for $A^{(2\alpha+1)}$. It is straightforward to prove [4] that this results in the fact that terms in (38) containing $F^{(2\alpha+2)}$, ($\alpha > 0$) are globally exact and only contribute to the index in a trivial way. Finally we find for the index corresponding to the Dirac operator (35)

$$\Delta = \frac{1}{(2\pi)^{d/2}} \int_{\text{space}} \mathrm{Tr} \exp \mathrm{i}F^{(2)} \left[\det \left(\frac{\mathrm{i}R_{ab}/2}{\sinh(\mathrm{i}R_{ab}/2)} \right) \right]^{1/2}. \tag{39}$$

In the cases that we investigated we found that the extra fields that can be added to the spinning particle model do not give any non-trivial contribution to the index. This means that if we couple a Dirac fermion not only to gravity and non-Abelian gauge fields but also to (non-Abelian) anti-symmetric tensor fields, no extra contributions to the axial anomaly will be found. This is in agreement with the results of Alvarez-Gaumé and Witten.

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